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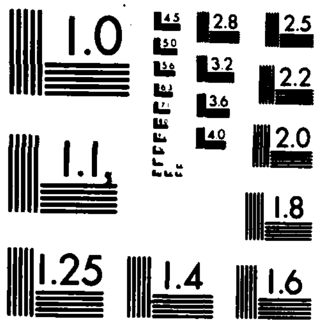

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Technical Report No. 392

SPECTRAL FORMULATION OF INTERNAL WAVE INTERACTION

by

COLIN SHEN AND GREG HOLLOWAY

Office of Naval Research
Contract No. N00014-80-C-0252

Reference M81-23
October, 1982

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Abstract

Fourier representations of the equations of motion for a rotating stratified fluid are obtained both in the two-dimensional and three-dimensional wave number space. It is shown that these spectral equations can be further transformed to reveal explicitly the interaction between waves and waves and between waves and eddies by decomposing the Fourier coefficients in terms of three basis functions, that correspond to the "upward" propagating and the "downward" propagating wave modes and the zero frequency horizontal eddy mode in the linear analysis. In the case of two-dimensional rotating stratified fluid motion, the transformation is made and full spectral equations up to second moments for these basis functions are derived.

Acknowledgement

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1. Remarks

Rotating stratified wave-turbulence interaction is a highly complex process, and one must continually explore and examine for the best approach to a satisfactory description of the process. In this report, we present a spectral formulation of the wave-turbulence interaction. Such a formulation has the advantage over the perturbative approach currently used in the literature (e.g. McComas and Bretherton, 1977) in that interaction equations can be derived for finite amplitude up to the Boussinesq approximation, whereas the perturbative approach by the very nature of the technique must assume small amplitude for the derivation. Our spectral formulation thus can potentially improve the description of the interaction, especially in the strong interaction regime.

The spectral equations of motion in this report are obtained in terms of three functions that represent separately the 'upward' propagating, the 'downward' propagating inertial gravity waves, and the zero frequency horizontal eddy current; Equations expressed in this manner have the advantage that the interaction between waves and between waves and eddies can be separated out and examined explicitly. In view that the equations may be of general interest, sufficiently detailed derivations are given. The derivations also serve as a reference for the spectral equations used in the recent numerical modelling study of nonlinear internal waves by Shen and Holloway (1982).

The interaction equations are derived below for a stratified rotating fluid system independent of one horizontal axis. This simpler two dimensional case is considered because it is most frequently numerically simulated, and the full detail of the interaction can be derived. The more difficult three dimensional case is discussed in a memo by Holloway which is attached as an appendix to this report.

The fluid system considered is unbounded, incompressible, uniformly rotating and uniformly stratified in the mean. The system satisfies the Boussinesq approximation; that is, fluid accelerations are small compared with gravity and vertical excursions are small compared with density scale height.

2. Equations of motion on a vertical plane

We shall briefly state the equations of motion from which the spectral equations for the two dimensional case are to be derived. The relevant quadratic integral invariants for this case will also be noted.

The coordinate system adopted here is the usual right-handed rectangular Cartesian coordinate system in which x and y denote the horizontal axes and z denotes the vertical axis; the direction of z is opposite to the local gravity $\underline{g}$. The local component of earth's rotation $\underline{f}$ is along z. In this system the two dimensional motions are assumed to be functions of x and z coordinates only. The full equations of motion which satisfy the Boussinesq approximation in two dimension are then

$$\partial_t u + u\partial_x u + w\partial_z u - fv = -\rho_o^{-1}\partial_x P + \nu\nabla^2 u \tag{2.1a}$$

$$\partial_t v + u\partial_x v + w\partial_z v + fu = \nu\nabla^2 v \tag{2.1b}$$

$$\partial_t w + u\partial_x w + w\partial_z w = -\rho_o^{-1}\partial_x P + \nu\nabla^2 w \tag{2.1c}$$

$$\partial_x u + \partial_z w = 0 \tag{2.1d}$$

$$\partial_t \rho' + u\partial_x \rho' + w\partial_z \rho' = Rw + D\nabla^2 \rho' \tag{2.1e}$$

where u,v,w are the velocity components along x,y,z, respectively. P is the pressure. ρ_o is the mean density. ρ' is the perturbation density defined by $\rho' = \rho - \rho_o(1 - Rz)$, where $R = |\partial_z\bar{\rho}|$ is a constant mean density gradient. ν is the viscosity and D is the diffusivity e.g., via thermal conduction.

For the derivation, it will be much simpler to work with a reduced set of the above equations. This is obtained by eliminating the pressure through cross differentiating (2.1a) and (2.1c) and by replacing u and w with a streamfunction ψ via. $u = \partial_z\psi$ and $w = -\partial_x\psi$. The result is

$$\partial_t \zeta - f\partial_z v - g\partial_x \rho' = J(\psi,\zeta) + \nu\nabla^2\zeta \tag{2.2a}$$

$$\partial_t v + f\partial_z \psi = J(\psi,v) + \nu\nabla^2 v \tag{2.2b}$$

$$\partial_t \rho' + R\partial_x \psi = J(\psi,\rho') + D\nabla^2 \rho' \tag{2.2c}$$

where $\zeta = \nabla^2\psi$ is the relative vorticity and $J(A,B) = \partial_x A \partial_y B - \partial_x B \partial_z A$ is the Jacobian. Thus the field is adequately described by three physical variables ψ, v, and ρ'.

In later spectral formulation, we shall consider only fluid region which has periodic boundary conditions. In such a region the net flux through the boundary is zero, and two quadratic integrals are invariant provided that dissipation and diffusion are absent. The first is the conservation of energy, $\partial_t \iint E \, dxdz = 0$, where

$$E = (|\nabla\psi|^2 + v^2 + N^2\rho'^2/R^2)/2 \qquad (2.3)$$

and $N^2 = gR/\rho_0$ is the Brunt-Vaisala frequency squared. The energy conservation is obtained by multiplying the three equations by ψ, v, and ρ', respectively, summing and then integrating. The other invariant quantity is the potential vorticity

$$PV = [(\partial_x v + f)\partial_z(\bar{\rho} + \rho') - \partial_z v \partial_x \rho'] \qquad (2.4)$$

which is obtained by taking ∂_x and ∂_z of (2.2b) alternately and by applying the same operation to (2.2c) and then forming the product shown in (2.4). This is a special case of the Ertel's theorm. These invariants are useful as diagnostics for numerical simulation study and also suggest useful second moment quantities for statistical study (see eq. 3.15 and Sec. 3-D). Lastly, we note the existence of a third invariant for the special case of $f = 0$. In this case, the quantity $\rho'\zeta$ is also conserved. This quadratic invariant can be obtained by multiplying (2.2a) and (2.2c) by ρ' and ζ, respectively and then integrating.

3. Derivation

The derivation will proceed in three steps: a) Fourier transform the equations of motion; b) obtain the basis functions for the Fourier coefficients; and c) construct the second moment spectral evolution equations. In step b, the basis functions refer to the functional representation of the two inertial gravity wave modes - i.e., the 'upward' and the 'downward' propagating modes - and the zero frequency current mode in the linear wave analysis. We choose such representation so that interactions between waves and eddies can be examined separately.

A. Fourier representation

The domain over which we obtain the Fourier expansion is assumed to have dimension $2\pi \times 2\pi$. The choice is for convenience since the dimension can always be rescaled later to the size of interest. Thus in terms of discrete set of wave vectors, Fourier expansions for ψ, v, ρ' are

$$\psi = \sum_{\underline{k}} \psi_{\underline{k}} \exp(i\underline{k}\cdot\underline{x}) \tag{3.1a}$$

$$v = \sum_{\underline{k}} v_{\underline{k}} \exp(i\underline{k}\cdot\underline{x}) \tag{3.1b}$$

$$\rho' = \sum_{\underline{k}} \rho'_{\underline{k}} \exp(i\underline{k}\cdot\underline{x}) \tag{3.1c}$$

with $\psi_{\underline{k}} = \psi^{*}_{-\underline{k}}$, etc., since the physical variables are real.

Substitution of the above into (2.2), the equations for Fourier coefficients are

$$\partial_t \psi_{\underline{k}} + \nu k^2 \psi_{\underline{k}} + i(f k_z v_{\underline{k}} + g k_x \rho'_{\underline{k}})/k^2 = \sum_{\underline{p}+\underline{q}=\underline{k}} (k_x p_y - k_y p_x)(q^2/k^2)\, \psi_{\underline{p}}\, \psi_{\underline{q}} \tag{3.2a}$$

$$\partial_t v_{\underline{k}} + \nu k^2 v_{\underline{k}} + i\, f k_z \psi_{\underline{k}} = \sum_{\underline{p}+\underline{q}=\underline{k}} (k_x p_y - k_y p_x)\, \psi_{\underline{p}}\, v_{\underline{q}} \tag{3.2b}$$

$$\partial_t \rho'_{\underline{k}} + D k^2 \rho'_{\underline{k}} + i\, R k_x \psi_{\underline{k}} = \sum_{\underline{p}+\underline{q}=\underline{k}} (k_x p_y - k_y p_x)\, \psi_{\underline{p}}\, \rho'_{\underline{q}} \tag{3.2c}$$

For the convenience of later discussion, the above in the matrix notation is

$$(W - \Lambda)\,\overline{X}_{\underline{k}} = \sum_{\underline{p}+\underline{q}=\underline{k}} N_{\underline{kpq}} : \overline{X}_{\underline{p}}\overline{X}_{\underline{q}} \tag{3.3}$$

where

$$\overline{X} = \begin{bmatrix} \psi_{\underline{k}} \\ v_{\underline{k}} \\ \rho'_{\underline{k}} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -\partial_t & 0 & 0 \\ 0 & -\partial_t & 0 \\ 0 & 0 & -\partial_t \end{bmatrix} \tag{3.4}$$

$$W = \begin{bmatrix} \nu k^2 & ifk_z/k^2 & igk_x/k^2 \\ ifk_z & \nu k^2 & 0 \\ iRk_x & 0 & Dk^2 \end{bmatrix}$$

and $N_{\underline{kpq}}$ is a 3x3x3 matrix whose only nonzero elements are

$$N_{111} = (k_x p_y - k_y p_x)(q^2/k^2)$$

$$N_{221} = k_x p_y - k_y p_x$$

$$N_{331} = k_x p_y - k_y p_x$$

B. The Basis Functions

We will obtain linear solutions to the equations (3.2), generalize the solutions to basis functions and then find the transformation between the Fourier coefficients of physical variables and the basis functions.

Removing the nonlinear terms from (3.2) and substituting the solution $\overline{X}_{\underline{k}} = \{\hat{\psi}_{\underline{k}}, \hat{v}_{\underline{k}}, \hat{\rho}_{\underline{k}}\} \exp(i\omega t)$ into the linear part of the equation, one has a eigenvalue problem for ω, in which the following determinant must vanish in order for the solution to exist,

$$\begin{vmatrix} i\omega + \nu k^2 & ifk_z/k^2 & igk_x/k^2 \\ ifk_z & i\omega + \nu k^2 & 0 \\ iRk_x & 0 & i\omega + Dk^2 \end{vmatrix} = 0 \ , \qquad (3.5)$$

that is, $(i\omega + \nu k^2)^2(i\omega + Dk^2) + (fk_z/k)^2(i\omega + Dk^2) + (Nk_x/k)^2(i\omega + \nu k^2) = 0.$

This equation is third order in ω which generally yields rather complicated expressions for the frequency except for the case, $D = \nu$. We will briefly indicate below the effect on frequency when $D \neq \nu$. For the remainder of the derivation, only the simple case $D = \nu$ will be considered. However the procedure used below for obtaining the basis functions remains general.

In the case of kinematic viscosity equal to diffusion, the solutions for the frequency are

$$\begin{aligned} \omega_1 &= i\nu k^2 \\ \omega_2 &= \omega_0 + i\nu k^2 \\ \omega_3 &= -\omega_0 + i\nu k^2 \end{aligned} \qquad (3.6)$$

with $\omega_0^2 = (f^2k_z^2 + N^2k_x^2)/k^2$. Thus the first solution is the diffusively damped zero frequency current mode. The other two solutions are frequencies for inertial gravity waves with dissipation. When $D \neq \nu$, a frequency shift from the natural frequency ω_0 and additional damping

are to be expected. We briefly indicate this by considering asymptotic behavior of ω near $\varepsilon = 0$, with $\varepsilon = (D - \nu)/\nu$. Let $Q = i\omega + \nu k^2$. Eq. (3.5) then becomes $Q^3 + \varepsilon\nu k^2 Q^2 + \omega_o^2 Q + \varepsilon\nu(fk_z)^2 = 0$. For $\varepsilon \ll 1$, this can be solved readily by perturbative technique. Letting $Q = Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \ldots$, the zeroth order solution has the three roots

$$Q_{01} = 0 \ , \quad Q_{02} = i\omega_o \ , \text{ and } \quad Q_{03} = -\, i\omega_o \quad .$$

To the first order, the three roots are

$$Q_{11} = -\,\nu(fk_z)^2/\omega_o^2$$

$$Q_{12} = -\,\nu(k_x N)^2/2\omega_o^2 \quad \text{and} \quad Q_{13} = Q_{12}$$

To the second order, the three roots are

$$Q_{21} = 0$$

$$Q_{22} = i(\nu k_x^2/2)(N/\omega_o)^2(\nu k^2/\omega_o)[1 + (3/4)(N/\omega_o)^2(k_x/k)^2]$$

$$\text{and} \quad Q_{23} = -\,Q_{22} \quad .$$

Thus for $D \neq \nu$ and $\varepsilon \ll 1$, the frequencies are

$$\omega_1 = i\nu[k^2 + \varepsilon k_z^2(f/\omega_o)^2] + \ldots$$

$$\omega_2 = \omega_o + \varepsilon^2(\nu k_x^2/2)(N/\omega_o)^2(\nu k^2/\omega_o)[1 + (3/4)(N/\omega_o)^2(k_x/k)^2]$$
$$+ i\nu[k^2 + \varepsilon k_x^2(N/\omega_o)^2/2 + \ldots$$

$$\text{and} \quad \omega_3 = -\,\mathrm{Real}(\omega_2) + \mathrm{Imag}(\omega_2)$$

These show that the natural frequency tends to increase when D and ν differ, and the dissipation increases or decreases slightly depending on the sign of ε.

Now returning to the case $D = \nu$ being considered for the derivation of basis functions, the dispersion relations are (3.6). The eigenvector for each frequency can be solved for and these are

$$\hat{\bar{X}}_{1,\underline{k}} = \{0,\ -gk_x/fk_z,\ 1\}\ \hat{h} \tag{3.7a}$$

$$\hat{\bar{X}}_{2,\underline{k}} = \{1,\ -fk_z/\omega_0,\ -Rk_x/\omega_0\}\ \hat{a}^- \tag{3.7b}$$

$$\hat{\bar{X}}_{3,\underline{k}} = \{1,\ fk_z/\omega_0,\ Rk_x/\omega_0\}\ \hat{a}^+ \tag{3.7c}$$

where the elements correspond to the elements in the vector $\hat{\bar{X}}_{\underline{k}} = \{\hat{\psi},\hat{v},\rho'\}_{\underline{k}}$ and $\hat{h}$, $\hat{a}^-$, $\hat{a}^+$ are arbitrary constants. The above three vectors are linearly independent. Hence, any solution of $\{\psi_{\underline{k}}, v_{\underline{k}}, \rho_{\underline{k}}\}$ is given as a linear combination of the three vectors

$$\bar{X}_{\underline{k}} = \hat{\bar{X}}_{1,\underline{k}}\ e^{i\omega_1 t} + \hat{\bar{X}}_{2,\underline{k}}\ e^{i\omega_2 t} + \hat{\bar{X}}_{3,\underline{k}}\ e^{i\omega_3 t} \tag{3.8}$$

At this point we may generalize to arbitrary t-dependence by denoting

$$a^+_{\underline{k}} = \hat{a}^+ \exp(i\omega_3 t),\ a^-_{\underline{k}} = \hat{a}^- \exp(i\omega_2 t),\ h_{\underline{k}} = \hat{h}\ \exp(i\omega_1 t)\ ,$$

where the first generalize function $a^+_{\underline{k}}$ may be thought as a 'upward' propagating wave mode since for a given $\underline{k}$ the frequency has opposite sign to $\underline{k}$. Similarly., $a^-_{\underline{k}}$ may be thought of a 'downward' propagating wave mode, and $h_{\underline{k}}$ a nonpropagating current mode. Thus, formally we have three basis functions representing three elementary modes of motion in a stratified rotating fluid. The transformation from the basis to the Fourier coefficients of the physical variables is readily obtained from (3.8). In matrix form the transformation is

$$\bar{X}_{\underline{k}} = M\ \bar{Y}_{\underline{k}}$$

$$\begin{bmatrix} \psi_{\underline{k}} \\ v_{\underline{k}} \\ \rho'_{\underline{k}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ fk_z/\omega_0 & -fk_z/\omega_0 & -gk_x/fk_z \\ Rk_x/\omega_0 & -Rk_x/\omega_0 & 1 \end{bmatrix} \begin{bmatrix} a^+_{\underline{k}} \\ a^-_{\underline{k}} \\ h_{\underline{k}} \end{bmatrix} \tag{3.9}$$

It follows that the inverse transformation from physical variables to the basis function is

$$Y_{\underline{k}} = M^{-1} X_{\underline{k}}$$

$$\begin{bmatrix} a^{+}_{\underline{k}} \\ a^{-}_{\underline{k}} \\ h_{\underline{k}} \end{bmatrix} = \begin{bmatrix} 1/2 & (fk_z)/(2k^2\omega_o) & (gk_x)/(2k^2\omega_o) \\ 1/2 & -(fk_z)/(2k^2\omega_o) & -(gk_x)/(2k^2\omega_o) \\ 0 & -(fk_z Rk_x)/(k\omega_o)^2 & (fk_z)^2/(k\omega_o)^2 \end{bmatrix} \begin{bmatrix} \psi_{\underline{k}} \\ v_{\underline{k}} \\ \rho'_{\underline{k}} \end{bmatrix} \tag{3.10}$$

Later calculation will be considerablly simplified if all the above variables are nondimensionalized. This can be achieved by letting

$$A^{+}_{\underline{k}} = \left[\frac{kR}{N}\, a^{+}_{\underline{k}}\right] \qquad A^{-}_{\underline{k}} = \left[\frac{kR}{N}\, a^{-}_{\underline{k}}\right] \qquad H_{\underline{k}} = \left[\frac{\omega_o k}{fk_z}\, h_{\underline{k}}\right]$$

$$\Psi_{\underline{k}} = \left[\frac{R^2}{N}\, \psi_{\underline{k}}\right] \qquad V_{\underline{k}} = \left[\frac{R}{N}\, v_{\underline{k}}\right] \qquad \ell_{\underline{k}} = \left[\frac{k}{R}\right] \tag{3.11}$$

$$\Omega_{\underline{k}} = (fk_z)/(k\omega_o) \qquad \eta_{\underline{k}} = (Nk_x)/(k\omega_o)$$

The foregoing transformation matrices in terms of dimensionless variables are

$$M^{-1} = \begin{pmatrix} \ell_{\underline{k}}/2 & \Omega_{\underline{k}}/2 & \eta_{\underline{k}}/2 \\ \ell_{\underline{k}}/2 & -\Omega_{\underline{k}}/2 & -\eta_{\underline{k}}/2 \\ 0 & -\eta_{\underline{k}} & \Omega_{\underline{k}} \end{pmatrix} \tag{3.12a}$$

$$M = \begin{pmatrix} 1/\ell_{\underline{k}} & 1/\ell_{\underline{k}} & 0 \\ \Omega_{\underline{k}} & -\Omega_{\underline{k}} & -\eta_{\underline{k}} \\ \eta_{\underline{k}} & -\eta_{\underline{k}} & \Omega_{\underline{k}} \end{pmatrix} \tag{3.12b}$$

C. Some identities:

For reference purpose, summarized here are the identities resulting from the Fourier transform.

The reality condition requires

$$\overset{*}{\Psi}_{\underline{k}} = \Psi_{-\underline{k}} \qquad \overset{*}{V}_{\underline{k}} = V_{-\underline{k}} \qquad \overset{*}{\rho}'_{\underline{k}} = \rho'_{-\underline{k}} \tag{3.13}$$

where * denotes the complex conjugate.

The transformation together with the reality condition require

$$\begin{aligned} \overset{*+}{A}_{\underline{k}} &= A^{-}_{-\underline{k}} \\ \overset{*-}{A}_{\underline{k}} &= A^{+}_{-\underline{k}} \\ \overset{*}{H}_{\underline{k}} &= -H_{-\underline{k}} \end{aligned} \tag{3.14}$$

The mean energy in terms of Fourier coefficients is

$$\begin{aligned} E_{\underline{k}} &= \ell^2_{\underline{k}}\, \Psi_{\underline{k}} \overset{*}{\Psi}_{\underline{k}} + V_{\underline{k}} \overset{*}{V}_{\underline{k}} + \rho'_{\underline{k}} \overset{*}{\rho}'_{\underline{k}} \\ &= 2(A^{+}_{\underline{k}} \overset{*+}{A}_{\underline{k}} + A^{-}_{\underline{k}} \overset{*-}{A}_{\underline{k}}) + H_{\underline{k}} \overset{*}{H}_{\underline{k}} \end{aligned} \tag{3.15}$$

Finally, for the dimensionless parameters we note

$$\begin{aligned} \ell_{-\underline{k}} &= \ell_{\underline{k}} \\ \Omega_{-\underline{k}} &= -\Omega_{\underline{k}} \\ \eta_{-\underline{k}} &= -\eta_{\underline{k}} \\ \Omega^2_{\underline{k}} + \eta^2_{\underline{k}} &= 1. \end{aligned} \tag{3.16}$$

D. Spectral Equations for the Basis Functions

Equations (3.3) for the Fourier coefficients can now be transformed into equations for the basis function. This is done simply by substituting $M\,\overline{Y}_{\underline{k}}$ for $\overline{X}_{\underline{k}}$ and then multiplying both sides of (3.3) by M^{-1}, i.e.,

$$M^{-1}(W - \Lambda)\, M\, \overline{Y}_{\underline{k}} = M^{-1} \sum_{\underline{p}+\underline{q}=\underline{k}} N_{\underline{kpq}} : \overline{X}_{\underline{p}} \overline{X}_{\underline{q}}$$

Using the dimensionless form of M and M^{-1} and rewriting the W matrix as

$$W = \begin{bmatrix} \mu^2_{\underline{k}} & i\omega_{\underline{k}}\Omega_{\underline{k}}/\ell_{\underline{k}} & i\omega_{\underline{k}}\eta_{\underline{k}}/\ell_{\underline{k}} \\ i\omega_{\underline{k}}\ell_{\underline{k}}\Omega_{\underline{k}} & \mu^2_{\underline{k}} & 0 \\ i\omega_{\underline{k}}\ell_{\underline{k}}\eta_{\underline{k}} & 0 & \mu^2_{\underline{k}} \end{bmatrix} \tag{3.17}$$

by applying the scaling in (3.11), the above matrix multiplication yields

$$\begin{bmatrix} \partial_t + (i\omega_{\underline{k}} + \mu^2_{\underline{k}}) & 0 & 0 \\ 0 & \partial_t + (-i\omega_{\underline{k}} + \mu^2_{\underline{k}}) & 0 \\ 0 & 0 & \partial_t + \mu^2_{\underline{k}} \end{bmatrix} \begin{bmatrix} A^+_{\underline{k}} \\ A^-_{\underline{k}} \\ H_{\underline{k}} \end{bmatrix} = M^{-1} \sum_{\underline{k}} N_{\underline{kpq}} : \overline{X}_{\underline{p}} \overline{X}_{\underline{q}} \tag{3.18}$$

where we have scaled $\omega_{\underline{k}} = \omega_o/N$ and $\mu_{\underline{k}} = \nu k^2/NR^2$. The nonlinear terms on the right hand side of (3.18) are

$$\text{for } A^+_{\underline{k}} :\quad 1/2 \sum_{\underline{p}+\underline{q}=\underline{k}} \ell_{\underline{k}}\ell_{\underline{p}}\theta_{\underline{k},\underline{p}}\Psi_{\underline{p}}[(\ell_{\underline{p}}/\ell_{\underline{q}})^2\Psi_{\underline{q}}\ell_{\underline{k}} + \Omega_{\underline{k}}V_{\underline{q}} + \eta_{\underline{k}}\rho'_{\underline{q}}] \tag{3.19a}$$

$$A^-_{\underline{k}} :\quad 1/2 \sum_{\underline{p}+\underline{q}=\underline{k}} \ell_{\underline{k}}\ell_{\underline{p}}\theta_{\underline{k},\underline{p}}\Psi_{\underline{p}}[(\ell_{\underline{p}}/\ell_{\underline{q}})^2\Psi_{\underline{q}}\ell_{\underline{k}} - \Omega_{\underline{k}}{}_{\underline{q}} - \eta_{\underline{k}}\rho'_{\underline{q}}] \tag{3.19b}$$

$$H_{\underline{k}} :\quad \sum_{\underline{p}+\underline{q}=\underline{k}} \ell_{\underline{k}}\ell_{\underline{p}}\theta_{k,p}\Psi_{\underline{p}}[-\eta_{\underline{k}}V_{\underline{q}} + \Omega_{\underline{k}}\rho'_{\underline{q}}] \tag{3.19c}$$

where $\theta_{\underline{k},\underline{p}}$ is the sine of the angle between the wave vectors $\underline{k}$ and $\underline{p}$.

Now substituting A^+, A^-, and H for Ψ, V, ρ' using the transform relation (3.12b) and expanding the nonlinear terms, we have for the $A^+_{\underline{k}}$ equation the following,

$$\partial_t A^+_{\underline{k}} + (i\omega_{\underline{k}} + \mu^2_{\underline{k}})\, A^+_{\underline{k}} = \tag{3.20}$$

$$1/2 \sum_{\underline{p}+\underline{q}=\underline{k}} \ell_{\underline{k}} \ell_{\underline{p}} \theta_{\underline{k},\underline{p}} \ell_{\underline{k}} (A^+_{\underline{p}} + A^-_{\underline{p}})\, [(\ell_{\underline{q}}/\ell_{\underline{k}}) + \Omega_{\underline{k}}\Omega_{\underline{q}} + \eta_{\underline{k}}\eta_{\underline{q}}]A^+_{\underline{q}}$$
$$+[(\ell_{\underline{q}}/\ell_{\underline{k}}) - \Omega_{\underline{k}}\Omega_{\underline{q}} - \eta_{\underline{k}}\eta_{\underline{q}}]A^-_{\underline{q}}$$
$$-[\Omega_{\underline{k}}\eta_{\underline{q}} - \Omega_{\underline{q}}\eta_{\underline{k}}]H_{\underline{q}}$$

The nonlinear terms can be simplified somewhat by defining a frequency vector,

$$\tilde{\omega}_{\underline{k}} = \{\Omega_{\underline{k}}\ ,\ \eta_{\underline{k}}\} \tag{3.21}$$

This allows rewriting

$$\Omega_{\underline{k}}\Omega_{\underline{q}} + \eta_{\underline{k}}\eta_{\underline{q}} = \tilde{\omega}_{\underline{k}}\cdot\tilde{\omega}_{\underline{q}} = \tilde{\phi}_{\underline{k},\underline{q}} \quad ; \quad \Omega_{\underline{k}}\eta_{\underline{q}} - \eta_{\underline{k}}\Omega_{\underline{q}} = \tilde{\theta}_{\underline{k},\underline{q}}$$

which are, respectively, the cosine and sine of the angle between the two frequency vectors. Further to reduce the number of terms, we introduce

$$\Gamma_{\underline{p}} = A^+_{\underline{p}} + A^-_{\underline{p}} \tag{3.22}$$

which is the sum of an upward going and a downward going wave of wave-number $\underline{p}$. *The final form of the* $A^+_{\underline{k}}$ equation is

$$\partial_t A^+_{\underline{k}} + (i\omega_{\underline{k}} + \mu^2_{\underline{k}})\, A^+_{\underline{k}} = \tag{3.23}$$

$$1/2 \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{\underline{k},\underline{pq}}\Gamma_{\underline{p}}A^+_{\underline{q}} - \alpha_{-\underline{k},\underline{pq}}\Gamma_{\underline{p}}A^-_{\underline{q}} - \gamma_{\underline{k},\underline{pq}}\Gamma_{\underline{p}}H_{\underline{q}}$$

where the interaction coefficients are

$$\alpha_{\underline{k},\underline{pq}} = \theta_{\underline{k},\underline{p}}(\ell_{\underline{q}} + \ell_{\underline{k}}\tilde{\phi}_{\underline{k},\underline{q}}) \tag{3.24a}$$

$$\gamma_{\underline{k},\underline{pq}} = \ell_{\underline{k}}\theta_{\underline{k},\underline{p}}\tilde{\theta}_{\underline{k},\underline{q}} \tag{3.24b}$$

Thus, we see that the interactions are determined by the ratio of the magnitudes of wave vectors, the angle between the wave vectors and the angle between the frequency vectors; Furthermore, the generation of A^+ waves arises from both wave-wave interaction and wave-eddy interaction.

In a similar manner the A^- wave equation can be obtained by expanding the nonlinear terms and the result is

$$\partial_t A^-_{\underline{k}} + (-i\omega_{\underline{k}} + \mu^2_{\underline{k}}) A^-_{\underline{k}} \tag{3.25}$$

$$= 1/2 \sum_{\underline{p}+\underline{q}=\underline{k}} -\alpha_{-\underline{k},\underline{pq}} \Gamma_{\underline{p}} A^+_{\underline{q}} + \alpha_{\underline{k},\underline{pq}} \Gamma_{\underline{p}} A^-_{\underline{q}} + \gamma_{\underline{k},\underline{pq}} \Gamma_{\underline{p}} H_{\underline{q}}$$

The third spectral equation, the current mode H equation, after expanding becomes

$$\partial_t H_{\underline{k}} + \mu^2_{\underline{k}} H_{\underline{k}} = \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{pq}} \Gamma_{\underline{p}} A^+_{\underline{q}} - \gamma_{\underline{k},\underline{pq}} \Gamma_{\underline{p}} A^-_{\underline{q}} + \beta_{\underline{k},\underline{pq}} \Gamma_{\underline{p}} H_{\underline{q}} \tag{3.26}$$

where the coefficient β is $\beta_{\underline{k},\underline{pq}} = \ell_{\underline{k}} \theta_{\underline{k},\underline{p}} \tilde{\phi}_{\underline{k},\underline{q}}$ and is related to the coefficient

$$\alpha_{\underline{k},\underline{pq}} = \ell_{\underline{q}} \theta_{\underline{k},\underline{p}} + \beta_{\underline{k},\underline{pq}} \quad . \tag{3.27}$$

D. Second and third moment spectral equations

In the foregoing the spectral interaction equations for A^+, A^-, and H are derived. As a final item, we will write down the governing equations for the second moment of these three functions in anticipation that these equations will be useful for a statistical treatment of rotating stratified wave-turbulence interaction. The closeure for these equations frequently requires the consideration of the third moment equations. For completeness, we will also record the details of the third moment equations at the end.

The three basis functions A^+, A^-, and H imply six different combinations of second moments among them. However, we will record only three moments here, namely, $\langle A^+_{\underline{k}}\, A^+_{\underline{m}} \rangle$, $\langle A^+_{\underline{k}}\, H_{\underline{m}} \rangle$, $\langle H_{\underline{k}}\, H_{\underline{m}} \rangle$, since the moment equations associated with the A^- function can be obtained by simply changing the sign of the interaction coefficients as will be indicated below. The angle brackets here denote ensemble average. Note that in the case of homogeneity which is often assumed in statistical study, the number of moments is reduced one more to only $\langle A^+_{\underline{k}}\, A^+_{\underline{k}} \rangle$ and $\langle H_{\underline{k}}\, H_{\underline{k}} \rangle$.

The second moment equation for $A^+_{\underline{k}}\, A^+_{\underline{m}}$ is obtained by forming the products $A^+_{\underline{k}}\, \partial_t A^+_{\underline{m}}$ and $A^+_{\underline{m}}\, \partial_t A^+_{\underline{k}}$. The two products are then added and averaged. The same procedure applies to the other two moment equations. The results are given below

$$
\begin{aligned}
&\partial_t \langle A^+_{\underline{k}}\, A^+_{\underline{m}} \rangle + [i(\omega_{\underline{k}} + \omega_{\underline{m}}) + \mu^2_{\underline{k}} + \mu^2_{\underline{m}}] \langle A^+_{\underline{k}}\, A^+_{\underline{m}} \rangle \qquad (3.28) \\
&\quad = \frac{1}{2} \Big\{ \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{m}} \rangle + \sum_{\underline{p}+\underline{q}=\underline{m}} \alpha_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{k}} \rangle \\
&\qquad - \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{-\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{m}} \rangle - \sum_{\underline{p}+\underline{q}=\underline{m}} \alpha_{-\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{k}} \rangle \\
&\qquad - \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{m}} \rangle - \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m}\,\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{m}} \rangle \Big\}
\end{aligned}
$$

$$\partial_t \langle A^+_{\underline{k}} H_{\underline{m}} \rangle + [i\omega_{\underline{k}} + \mu^2_{\underline{k}} + \mu^2_{\underline{m}}] \langle A^+_{\underline{k}} H_{\underline{m}} \rangle \tag{3.29}$$

$$= \frac{1}{2} \{ \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{m}} \rangle \} + \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{k}} \rangle$$

$$- \frac{1}{2} \{ \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{-\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{m}} \rangle \} - \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{k}} \rangle$$

$$- \frac{1}{2} \{ \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{m}} \rangle \} + \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{k}} \rangle$$

$$\partial_t \langle H_{\underline{k}} H_{\underline{m}} \rangle + [\mu^2_{\underline{k}} + \mu^2_{\underline{m}}] \langle H_{\underline{k}} H_{\underline{m}} \rangle \tag{3.30}$$

$$= \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{m}} \rangle + \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{k}} \rangle$$

$$- \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{m}} \rangle - \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{m}} \rangle$$

$$+ \sum_{\underline{p}+\underline{q}=\underline{k}} \beta_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{m}} \rangle + \sum_{\underline{p}+\underline{q}=\underline{m}} \beta_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{k}} \rangle$$

Now note that the spectral equations (3.23) for A^+ and (3.25) for A^- differ only in the sign of their coefficients and subscripts. Thus the above second moment equation for A^+ can be converted to one for A^- by applying this simple rule: Change the superscript of the A function(s) in the time dependent term to negative and then note the subscript associated with that function. In the remaining terms, change the sign of the frequency associated with that subscript, the sign of the interaction coefficients containing that subscript, the sign of the subscript itself in the coefficients, and the sign of the superscript of that particular A function throughout. This completes the general second moment equations. Among the second moment equations, of particular interest are moment equations describing the spectral energy evolution. Eq. (3.15) gives the expression of spectral energy. Since $A^+_{\underline{k}} A^{*+}_{\underline{k}} = A^+_{\underline{k}} A^-_{-\underline{k}}$ the above equation (3.28) can be immediately converted to one describing the spectral energy of $A^+_{\underline{k}}$ mode by applying the just stated conversion rule. The left-hand side of (3.28) is irreducible, but the right-hand

side is simply, for $A^+_{\underline{k}} \overset{*}{A}{}^+_{\underline{k}}$,

$$\partial_t \langle A^+_{\underline{k}} A^-_{-\underline{k}} \rangle + 2\mu^2_{\underline{k}} \langle A^+_{\underline{k}} A^-_{-\underline{k}} \rangle$$

for $A^-_{\underline{k}} \overset{*}{A}{}^-_{\underline{k}}$

$$\partial_t \langle A^-_{\underline{k}} A^+_{-\underline{k}} \rangle + 2\mu_{\underline{k}} \langle A^-_{\underline{k}} A^+_{-\underline{k}} \rangle$$

and for $H_{\underline{k}} \overset{*}{H}_{\underline{k}}$

$$-\partial_t \langle H_{\underline{k}} H_{-\underline{k}} \rangle - 2\mu_{\underline{k}} \langle H_{\underline{k}} H_{-\underline{k}} \rangle$$

The right-hand side of (3.28) to (3.30) contained third moment quantities. We conclude this report by listing four third moment equations for different combinations of A^+ and H basis functions. Moment equations involving A^- functions again can be deduced by changing signs according the rule given earlier.

$$\begin{aligned}
&\partial_t \langle A^+_{\underline{k}} A^+_{\underline{m}} A^+_{\underline{n}} \rangle + [i(\omega_{\underline{k}} + \omega_{\underline{m}} + \omega_{\underline{n}}) + \mu^2_{\underline{k}} + \mu^2_{\underline{m}} + \mu^2_{\underline{n}}] \langle A^+_{\underline{k}} A^+_{\underline{m}} A^+_{\underline{n}} \rangle \\
&= \frac{1}{2} \Big\{ \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{m}} A^+_{\underline{n}} \rangle - \alpha_{-\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{m}} A^+_{\underline{n}} \rangle - \gamma_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{m}} A^+_{\underline{n}} \rangle \\
&\quad + \sum_{\underline{p}+\underline{q}=\underline{m}} \alpha_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{n}} \rangle - \alpha_{-\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{n}} \rangle - \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{n}} \rangle \\
&\quad \sum_{\underline{p}+\underline{q}=\underline{n}} \alpha_{\underline{n},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{m}} \rangle - \alpha_{-\underline{n},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{m}} \rangle - \gamma_{\underline{n},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{m}} \rangle \Big\}
\end{aligned}$$

$$\begin{aligned}
&\partial_t \langle A^+_{\underline{k}} H_{\underline{m}} A^+_{\underline{n}} \rangle + [i(\omega_{\underline{k}} + \omega_{\underline{n}}) \qquad + \mu^2_{\underline{k}} + \mu^2_{\underline{m}} + \mu^2_{\underline{n}}] \langle A^+_{\underline{k}} H_{\underline{m}} A^+_{\underline{n}} \rangle \\
&= \frac{1}{2} \Big\{ \sum_{\underline{p}+\underline{q}=\underline{k}} \alpha_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{m}} A^+_{\underline{n}} \rangle - \alpha_{-\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{m}} A^+_{\underline{n}} \rangle - \gamma_{\underline{k},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{m}} A^+_{\underline{n}} \rangle \Big\} \\
&\quad + \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{n}} \rangle - \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{n}} \rangle + \gamma_{\underline{m},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{k}} A^+_{\underline{n}} \rangle \\
&\quad + \frac{1}{2} \Big\{ \sum_{\underline{p}+\underline{q}=\underline{n}} \alpha_{\underline{n},\underline{pq}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} A^+_{\underline{k}} H_{\underline{m}} \rangle - \alpha_{-\underline{n},\underline{pq}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} A^+_{\underline{k}} H_{\underline{m}} \rangle - \gamma_{\underline{n},\underline{pq}} \langle \Gamma_{\underline{p}} H_{\underline{q}} A^+_{\underline{k}} H_{\underline{m}} \rangle \Big\}
\end{aligned}$$

$$\partial_t \langle H_{\underline{k}} H_{\underline{m}} A^+_{\underline{n}} \rangle + [i\omega_{\underline{n}} + \mu^2_{\underline{k}} + \mu^2_{\underline{m}} + \mu^2_{\underline{n}}] \langle H_{\underline{k}} H_{\underline{m}} A^+_{\underline{n}} \rangle$$

$$= \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{m}} A^+_{\underline{n}} \rangle - \gamma_{\underline{k},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{m}} A^+_{\underline{n}} \rangle + \beta_{\underline{k},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{m}} A^+_{\underline{n}} \rangle$$

$$+ \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{k}} A^+_{\underline{n}} \rangle - \gamma_{\underline{m},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{k}} A^+_{\underline{n}} \rangle + \beta_{\underline{m},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{k}} A^+_{\underline{n}} \rangle$$

$$+ \frac{1}{2} \left\{ \sum_{\underline{p}+\underline{q}=\underline{n}} \alpha_{\underline{n},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{k}} H_{\underline{m}} \rangle - \alpha_{-\underline{n},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{k}} H_{\underline{m}} \rangle - \gamma_{\underline{n},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{m}} H_{\underline{n}} \rangle \right\}$$

$$\partial_t \langle H_{\underline{k}} H_{\underline{m}} H_{\underline{n}} \rangle + [\mu^2_{\underline{k}} + \mu^2_{\underline{m}} + \mu^2_{\underline{n}}] \langle H_{\underline{k}} H_{\underline{m}} H_{\underline{n}} \rangle$$

$$= \sum_{\underline{p}+\underline{q}=\underline{k}} \gamma_{\underline{k},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{m}} H_{\underline{n}} \rangle - \gamma_{\underline{k},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{m}} H_{\underline{n}} \rangle + \beta_{\underline{k},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{m}} H_{\underline{n}} \rangle$$

$$+ \sum_{\underline{p}+\underline{q}=\underline{m}} \gamma_{\underline{m},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{k}} H_{\underline{n}} \rangle - \gamma_{\underline{m},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{k}} H_{\underline{n}} \rangle + \beta_{\underline{m},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{k}} H_{\underline{n}} \rangle$$

$$+ \sum_{\underline{p}+\underline{q}=\underline{n}} \gamma_{\underline{n},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^+_{\underline{q}} H_{\underline{k}} H_{\underline{m}} \rangle - \gamma_{\underline{n},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} A^-_{\underline{q}} H_{\underline{k}} H_{\underline{m}} \rangle + \beta_{\underline{n},\underline{p}\underline{q}} \langle \Gamma_{\underline{p}} H_{\underline{q}} H_{\underline{k}} H_{\underline{m}} \rangle$$

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APPENDIX

Flow Research Technical Memo No. 236

To: Jim Riley, Mike Weissman, Ralph Metcalfe Date: April 1981

From: Greg Holloway

Subject: "Basis functions for analysis of stratified, rotating turbulence"

The object of this note is to set out the equations of motion for an unbounded, incompressible fluid which is uniformly rotating and uniformly stratified in the mean. Especially important is that the derivation remain fully valid for finite amplitude up to the Boussinesq approximation. We require such a confident basis before proceeding toward a statistical dynamical theory for the evolution of internal wave/stratified turbulence fields in oceans, atmospheres and laboratories. Calculations which follow are "straightforward but tedious" as is appropriate when laying a foundation. Sufficient detail is given so that no significant algebraic effort has been left to the reader's imagination.

1. Derivation of Interaction Equations

The instantaneous fluid state may consist of finite amplitude departures from the mean state which we seek to describe in a non-perturbative, complete way. In fact, we will make the Boussinesq approximation on the assumption that fluid accelerations are small compared with gravity and that vertical excursions are small compared with the density scale height. Then we begin by writing "usual" field equations for velocity $\underline{u}$, density deviation ρ and pressure p.

$$\partial_t \underline{u} + \underline{f} \times \underline{u} + \underline{u} \cdot \underline{\nabla} \underline{u} + \underline{\nabla} p + \underline{g}\rho - \nu\nabla^2 \underline{u} = 0 \qquad (1.a)$$

$$\partial_t \rho + \underline{u} \cdot \underline{\nabla}\rho - Ru_3 - \gamma \nabla^2 \rho = 0 \qquad (1.b)$$

$$\underline{\nabla} \cdot \underline{u} = 0 \qquad (1.c)$$

Here ρ is departure from a reference density $\rho_o(x_3) = 1 - Rx_3$ with R a positive constant and $gR = N^2$, the squared buoyancy frequency where we have nondimensionalized density by the mean density at the mean depth which is taken as $x_3 = 0$. We take $\underline{f}$ parallel to $\underline{g}$ upward along x_3. ν is viscosity while γ is a density diffusivity resulting, say, from thermal conduction in a thermally stratified fluid. We will consider this to be the only source of stratification and so will not consider double diffusive phenomena.

An important aside: The present formulation is quite different from a usual way of setting up the internal wave interaction equations. That method considers a field of fluid "particle" displacements $\underline{\varepsilon}(\underline{x},0)\equiv\underline{0}$ or to some "equilibrium" position $\underline{\varepsilon}_o(\underline{x},t)$. The two concepts can be confused, i.e., equivalenced. Diffusion is not considered. Interaction equations are then obtained as an infinite power series in the "small" displacements $|\underline{\xi}|$. Basis functions obtained from this series do not appear to be capable of a complete description of the possible fluid states. I feel this approach can only be useful for very small amplitude, non-dissipative waves.

Returning to the present derivation, we can evaluate pressure from the divergence of (1.a):

$$\frac{\partial}{\partial x_j}\varepsilon_{j3\ell} f u_\ell + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell} u_\ell u_j + \frac{\partial^2}{\partial x_j \partial x_j} P + g\frac{\partial}{\partial x_3}\rho = 0 \qquad (2)$$

where $\varepsilon_{j\ell m}$ is the unit alternating tensor and we sum on repeated indices. Next let all deviations from the mean state be constrained to be periodic over some large length scale, which may go to infinity. Nondimensionalizing that length scale to a volume of 2π, transform from the continuous spatial variable $\underline{x}$ to an infinite, discrete set of wavevectors $\underline{k}$ on which we denote Fourier coefficients of $\underline{u}$, p and ρ by

$$(\underline{u}, p, \rho) = \sum_{\underline{k}} (\underline{u}_{\underline{k}}, p_{\underline{k}}, \rho_{\underline{k}})\, e^{i\underline{k}\cdot\underline{x}} \qquad (3)$$

with components of $\underline{u}_{\underline{k}}$ denoted $u_{j,\underline{k}}$.

From (2)

$$P_{\underline{k}} = \frac{ik_j \varepsilon_{j3\ell}}{k^2} fu_{\ell,\underline{k}} - \frac{k_j k_\ell}{k^2} \sum_{\underline{p}+\underline{q}=\underline{k}} u_{\ell,\underline{p}} u_{\ell,\underline{q}} + \frac{igk_3}{k^2} \rho_{\underline{k}} \tag{4}$$

where $k^2 = \underline{k}\cdot\underline{k}$. Substitute this into the transform of (1.a) to obtain

$$\partial_t u_{j,\underline{k}} + \varepsilon_{j3\ell} fu_{\ell,\underline{k}} + ik_\ell \sum u_{\ell,\underline{p}} u_{j,\underline{q}}$$

$$+ ik_j \left[i \frac{k_m \varepsilon_{m3\ell} f}{k^2} u_{\ell,\underline{k}} - \frac{k_\ell k_m}{k^2} \sum u_{\ell,\underline{p}} u_{m,\underline{q}} \right.$$

$$\left. + \frac{igk_3}{k^2} \rho_{\underline{k}} \right] + g\rho_{\underline{k}} \delta_{j3} + \nu k^2 u_{j,\underline{k}} = 0$$

Combine some terms:

$$\varepsilon_{j3\ell} fu_{\ell,\underline{k}} - \frac{k_j k_m}{k^2} \varepsilon_{m3\ell} fu_{\ell,\underline{k}}$$

$$= \varepsilon_{m3\ell} fu_{\ell,\underline{k}} F_{jm,\underline{k}}$$

where

$$F_{jm,\underline{k}} = \delta_{jm} - \frac{k_j k_m}{k^2} \tag{5}$$

Likewise

$$ik_\ell \sum u_{\ell,\underline{p}} u_{j,\underline{q}} - i \frac{k_j k_\ell k_m}{k^2} \sum u_{\ell,\underline{p}} u_{m,\underline{q}}$$

$$= ik_\ell \left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) \sum u_{\ell,\underline{p}} u_{m,\underline{q}}$$

$$= ik_\ell F_{jm,\underline{k}} \sum u_{\ell,\underline{p}} u_{m,\underline{q}}$$

$$= iM_{j\ell m,\underline{k}} \sum_{\underline{p}+\underline{q}=\underline{k}} u_{\ell,\underline{p}}\, u_{m,\underline{q}}$$

where for symmetry we have introduced

$$M_{j\ell m} = \frac{1}{2}\left(k_\ell F_{jm} + k_m F_{j\ell}\right) \tag{6}$$

Now (1.a) can be rewritten

$$\partial_t u_{j,\underline{k}} + \varepsilon_{m3\ell} f u_{\ell,\underline{k}} F_{jm,\underline{k}} + g\rho_{\underline{x}} F_{j3,\underline{k}} \tag{7.a}$$

$$+ iM_{j\ell m,\underline{k}} \sum_{\underline{p}+\underline{q}=\underline{k}} u_{\ell,\underline{p}}\, u_{m,\underline{q}} + \nu k^2 u_{j,\underline{k}} = 0$$

while (1.b) and (1.c) are

$$\partial_t \rho_{\underline{k}} + ik_j \sum_{\underline{p}+\underline{q}=\underline{k}} u_{j\underline{p}}\, \rho_{\underline{q}} - R\delta_{j3}\, u_{j,\underline{k}} + \gamma k^2 \rho_{\underline{k}} = 0 \tag{7.b}$$

$$k_j u_{j,\underline{k}} = 0 \tag{7.c}$$

Equations (7.a,b) constitute four equations in four unknowns $\underline{u}_{\underline{k}}$, $\rho_{\underline{k}}$ having eliminated pressure. We may still use incompressibility (7.c) to eliminate one component of velocity. To do so, define a set of unit vectors for each $\underline{k}$:

$$\begin{aligned} \underline{e}^1_{\underline{k}} &= \underline{k} \times \underline{e}^3 \;/\; |\,\underline{k} \times \underline{e}^3\,| \\ \underline{e}^2_{\underline{k}} &= \underline{e}^3 \times (\,\underline{k} \times \underline{e}^3\,) \;/\; |\,\underline{e}^3 \times (\,\underline{k} \times \underline{e}^3\,)\,| \\ \underline{e}^3_{\underline{k}} &= \underline{\nabla}\, x_3 \end{aligned} \tag{8}$$

With respect to these unit vectors, define new velocity components

$$v_{\underline{k}} = \underline{u}_{\underline{k}} \cdot \underline{e}^1_{\underline{k}}$$

$$\tilde{u}_{\underline{k}} = \underline{u}_{\underline{k}} \cdot \underline{e}^{2}_{\underline{k}} \tag{9}$$

$$w_{\underline{k}} = \underline{u}_{\underline{k}} \cdot \underline{e}^{3}_{\underline{k}}$$

The horizontal component of $\underline{k}_h$ is $k_h = \underline{k} \cdot \underline{e}^2_k$. It will be convenient to denote ratios of k_h, k_3 and k as

$$\alpha = k_h / k \;, \; \beta = k_3 / k \;, \; \gamma = k_3 / k_h \tag{10}$$

whence we eliminate $\tilde{u}_{\underline{k}}$ by incompressibility

$$\tilde{u}_{\underline{k}} = - c\,(\underline{k})\, w_{\underline{k}} \tag{11}$$

The next object is to rewrite (7.a,b) in terms of $w_{\underline{k}}$, $v_{\underline{k}}$, $\rho_{\underline{k}}$. For the moment we will not worry about the nonlinear terms. Then we may consider a single wave vector $\underline{k}$ and so, for the moment, omit the $\underline{k}$ subscripts as being implied. $u_{\underline{k}}$ can be written

$$\underline{u} = v\underline{e}^1 - \sigma w\underline{e}^2 + w\underline{e}^3$$

We will need $\underline{e}^1$, $\underline{e}^2$, $\underline{e}^3$ in the original Cartesian coordinate system. Their components are

$$\begin{aligned} e^1_s &= \varepsilon_{sj3}\, k_j / k_h \\ e^2_s &= k_s / k_h \\ e^3_j &= \delta_{j3} \end{aligned} \tag{12}$$

where $s = 1,2$ and $k_h^2 = k_1^2 + k_2^2$.

Take first the $j = 3$ component of (7.a). Expand the term

$$\varepsilon_{m3\ell}\, f u_\ell\, F_{3m}$$

$$= \varepsilon_{m3\ell}\, f(\nu\varepsilon_{\ell p3}\, k_p - \sigma w k_\ell)\, (\delta_{3m} - \frac{k_3 k_m}{k^2}) \,/\, k_h$$

$$= - f v \frac{k_3 k_h}{k^2}$$

Denoting the nonlinear term only symbolically as $iMuu$, the $j = 3$ component of (7.a) is

$$(\partial_t + \nu k^2)\, w - f\alpha\beta v + g\alpha^2 \rho = - i(M\, uu)_3 \tag{13}$$

To obtain an equation for v , contract (7.a) with $\underline{e}^1$. This is

$$\varepsilon_{sj3} \frac{k_j}{k_h} \Big[(\partial_t + \nu k^2)\, u_s + \varepsilon_{m3\ell}\, f u_\ell\, F_{sm}$$

$$+ g\rho F_{s3} \Big] = - i \varepsilon_{sj3} \frac{k_j}{k_h} (M\, uu)_s$$

Expanding and collecting, obtain

$$\varepsilon_{sj3} \frac{k_j}{k_h} (\partial_t + \nu k^2)\, (v\varepsilon_{sm3}\, k_m - \sigma w k_s)$$

$$= \varepsilon_{sj3}\, \varepsilon_{sm3} \frac{k_j k_m}{k_h^2} (\partial_t + \nu k^2)\, v$$

$$= (\partial_t + \nu k^2)\, v$$

$$\varepsilon_{sj3} \frac{k_j}{k_h^2}\, \varepsilon_{m3\ell}\, f(\nu\varepsilon_{\ell p3}\, k_p - \sigma w k_\ell)\, (\delta_{sm} - \frac{k_s k_m}{k^2})$$

$$= -\varepsilon_{sj3}\, \varepsilon_{m3\ell}\, f\sigma w \frac{k_j k_\ell}{k_h^2}\, \delta_{sm}$$

$$= f\sigma w$$

$$\varepsilon_{sj3}\, k_j\, F_{s3} = 0$$

So,

$$(\partial_t + \nu k^2)v + f\alpha w = - i\varepsilon_{sj3} \frac{k_j}{k_h} (M\ uu)_s \tag{14}$$

Finally, from (7.b)

$$(\partial_t + \gamma k^2)\rho - Rw = - ik_j\ (u\rho)_j \tag{15}$$

where, again, we write the nonlinear terms symbolically and will worry about this later.

Gathering (13), (14) and (15), the system is

$$\begin{bmatrix} \partial_t + \nu k^2 & - f\alpha\beta & g\alpha^2 \\ f\alpha & \partial_t + \nu k^2 & 0 \\ -R & 0 & \partial_t + \gamma k^2 \end{bmatrix} \begin{bmatrix} w \\ v \\ \rho \end{bmatrix} = \begin{bmatrix} - i\ (M\ uu) \\ - i\varepsilon\frac{k}{k_h}\ (M\ uu) \\ - ik\ (u\rho) \end{bmatrix} \tag{16}$$

Now we can check that for linearized waves with $\nu = \gamma = 0$, solutions proportional to $e^{i\omega_o t}$ may be found when the determinant of the array in (16) vanishes. This yields

$$\omega_o^2 = N^2\alpha^2 + f^2\beta^2 \tag{17}$$

as well as a solution for $\omega_o = 0$. We have worked very hard to find linearized internal waves, but at least this tends to check our algebra! In fact (16) is valuable because, with a little more work, we could evaluate completely and exactly the nonlinear terms on the right side. This is in contrast to recent wave-wave interaction theories for which nonlinear terms are given by an infinite power series which cannot be evaluated exactly for any nonzero amplitude of the wave fields.

A quick overview of where we are and where we are going: A minimal statement of the dynamics given by (1) requires three equations in three unknowns, as seen in (16). If more unknowns are present, there is some diagnostic relation among them. If fewer unknowns are present, the description is incomplete, as recent wave interaction theories may be.

2. A New Basis

For unknowns we have considered w, v and ρ at each $\underline{k}$. Call these a "primitive" basis $\underline{y}_{\underline{k}} = (w, v, \rho)_{\underline{k}}$. In terms of $\underline{y}$, (16) has the form

$$(\underline{\underline{I}}\, \partial_t + \underline{\underline{L}})\ \underline{y} = i \sum \underline{\underline{N}} : \underline{y}\, \underline{y}$$

where $\underline{\underline{I}}$ is an identity and $\underline{\underline{L}}$ and $\underline{\underline{N}}$ are linear operators. Other bases are possible, in particular a "wave" basis $\underline{z}_{\underline{k}} = (a^+, a^-, h)_{\underline{k}}$, which will satisfy an equation like that for $\underline{y}$ but with a diagonal operator $\underline{\underline{L}}_o$ in the nondissipative case.

Were we to solve for the time evolution of all $\underline{y}_{\underline{k}}$ (or $\underline{z}_{\underline{k}}$), this would give us the history in detail of the flow field. We suppose that is neither practical nor especially useful. Rather we imagine an ensemble of statistically similar flows and we seek to describe the ensemble average (denoted $< \cdot >$) moments of the flow. Suppose there is no ensemble mean motion, hence $<\underline{y}_{\underline{k}}> = 0$. Interest then turns to the second moment tensors

$$\underline{\underline{Y}}_{\underline{k}} = <\underline{y}_{\underline{k}}\ \underline{y}_{\underline{k}}^*> \quad \text{or} \quad \underline{\underline{Z}}_{\underline{k}} = <\underline{z}_{\underline{k}}\ \underline{z}_{\underline{k}}^*>$$

Some of the components of $\underline{\underline{Y}}_{\underline{k}}$ include density variance, $< \rho\rho^* >_{\underline{k}}$; vertical mass flux, Real $[< \rho w^* >_{\underline{k}}]$; and enstrophy,

$$<(\underline{\nabla} \times \underline{u})_3^2>_{\underline{k}} = k_h^2 <vv^*>_{\underline{k}},$$

while total kinetic energy is

$$KE = (1 + \alpha^2) <ww^*>_{\underline{k}} + <vv^*>_{\underline{k}} \quad .$$

Gravitational potential energy is related to $< \rho\rho^* >_{\underline{k}}$ as

$$PE = < \rho\rho^* >_{\underline{k}}\ N^2 / R^2 \quad .$$

Reality of ρ, w and v in physical space is expressed by

$$y_{\underline{k}}^* = y_{-\underline{k}} \quad .$$

Consequently, e.g. $\langle\rho\rho^*\rangle_{\underline{k}} = \langle\rho\rho^*\rangle_{-\underline{k}}$.

Next let us explicitly obtain the wave bases $\underline{z}_{\underline{k}}$, omitting dissipation for the moment, i.e., $\nu = \gamma = 0$. From (16) see $(\partial_{tt} + \omega_o^2)\, w_{\underline{k}} = 0$. So obtain a^+ and a^- as

$$\omega_o a^{\pm} = (\partial_t \mp i\omega_o)\, w$$

$$= f\alpha\beta v - g\lambda^2 \rho \mp i\omega_o w$$

which then satisfy $(\partial_t \pm i\omega_o)\, a^{\pm} = 0$. The third basis component, denoted $h_{\underline{k}}$, is non-propagating and can be seen from (16)

$$\partial_t \left(v + \frac{f\sigma}{R}\rho\right) = 0$$

so $h = v - \frac{f\sigma}{R}\rho$. Together the transform is

$$\begin{bmatrix} -i\omega_o & f\alpha\beta & -g\lambda^2 \\ i\omega_o & f\alpha\beta & -g\lambda^2 \\ 0 & 1 & \frac{f\sigma}{R} \end{bmatrix} \begin{bmatrix} w \\ v \\ \rho \end{bmatrix}_{\underline{k}} = \begin{bmatrix} a^+ \\ a^- \\ h \end{bmatrix}_{\underline{k}} \tag{18}$$

such that the linear, inviscid system is given by a diagonal operator

$$\begin{bmatrix} \partial_t + i\omega_o & 0 & 0 \\ 0 & \partial_t - i\omega_o & 0 \\ 0 & 0 & \partial_t \end{bmatrix} \begin{bmatrix} a^+ \\ a^- \\ h \end{bmatrix}_{\underline{k}} = \underline{0} \tag{19}$$

Let us just note the inverse transform:

$$\begin{bmatrix} i/2 & -i/2 & 0 \\ \frac{f\sigma}{2\omega_o} & \frac{f\sigma}{2\omega_o} & \frac{N^2\alpha^2}{\omega_o^2} \\ -\frac{R}{2\omega_o} & -\frac{R}{2\omega_o} & \frac{Rf\alpha^2}{\omega_o^2} \end{bmatrix} \begin{bmatrix} a^+ \\ a^- \\ h \end{bmatrix}_{\underline{k}} = \begin{bmatrix} w \\ v \\ \rho \end{bmatrix}_{\underline{k}} \tag{20}$$

Physically, what are a^+, a^- and h ? The propagating modes a^+ and a^- can be termed the "upward" and "downward" propagating inertial-gravity waves. In terms of vertical phase propagation, the linearized, inviscid waves are described by

$$a^{\pm}(t)\, e^{ik_3x_3} \propto e^{i(\pm\omega_o t + k_3x_3)}$$

so that with x_3 upward we have a^+ the upward going wave in the sense of phase speed. Recall for these waves that the vertical component of group speed is of sign opposite the phase speed so that energy, say, associated with a^+ is downward going.

Perhaps more interesting are the h modes. To characterize these consider the pure case when $a^+ = a^- = 0$. Then from (20), or from the simultaneous conditions $a^+ = a^- = 0$, we see that pure h modes are given by

$$f\sigma v = g\rho \quad , \quad w = 0 \tag{21}$$

i.e., the horizontal velocity field is geostrophically balanced by variations in the density field. When we expand the right hand side of (16) in wave bases $\underline{z}_{\underline{p}}\, \underline{z}_{\underline{q}}$, then if we retain only $\underline{h}_{\underline{p}}\, \underline{h}_{\underline{q}}$ components, we will express "quasi-geostrophic turbulence". However, we will also see the transfer from quasi-geostrophic motion into the inertial-gravity waves. When inertial-gravity waves are excited, we may also see energy transfer from the waves to the quasi-geostrophic field. This is most tantalizing, suggesting a possible synthesis among quasi-geostrophic turbulence, internal waves and stratified, three-dimensional turbulence.

While it is generally acceptable to refer to h-modes as geostrophic modes, two other specific interpretations are interesting. Firstly, when $f = 0$, $k_h \neq 0$ then $h = v$ while v drops out of $a^{\pm}$. Vertical

vorticity $(\nabla \times \underline{u})_{3,\underline{k}} = k_h v$ is then given solely by h-modes which we would now be inclined to call "vortex modes" although they are the $f \to 0$ limit of geostrophic moes. Are these the horizontal pancake eddies left behind in wakes after more three-dimensional disturbances subside -- perhaps by radiation in the $a^{\pm}$ modes? Do nonlinearities in (16) lead to transfer of energy from h-modes to radiating $a^{\pm}$ modes during wake collapse?

A second point when $f = 0$ is the case $k_h \to 0$. In this limit, i.e., vertical $\underline{k}$, ρ may also contribute to h since $f\sigma$ is undefined. This vertical variation of horizontally uniform density is termed "density finestructure". The waves-finestructure interaction is fully included in the present analysis.

Another case $f \neq 0$, $k_h \to 0$, corresponding to density finestructure with rotation present appears to be troublesome insofar as $h \to \infty$. Probably it would have been more appropriate to define h differently as a variable $\tilde{h}$ for present purposes.

$$\tilde{h} = \alpha h = \alpha v + \frac{f\beta}{R} \rho \tag{22}$$

recognizing that the dual limit $f \to o$, $k_h \to o$ is special. $\tilde{h}$ like $a^{\pm}$ have been defined so that $\underline{z}_{\underline{k}} \underline{z}_{-\underline{k}}$ has dimensions of energy.

This is as far as seems useful to go in developing and discussing the different bases for the problem. What is important is that we retain a complete basis for describing the possible fluid motions. In particular we do not discard the geostrophic modes (or "vortex modes" or "density finestructure") as has been the rule with previous internal wave-wave interaction studies. Rather we view as very exciting the possible exchanges among h and $a^{\pm}$ modes on all scales. To see these we turn to expanding the right side of (16) both in primitive bases $\underline{y}_{\underline{k}}$ and in wave bases $\underline{z}_{\underline{k}}$.

3. Expansion of the Nonlinear Terms

Firstly in primitive bases we develop the right side of (13):

$$-iM_{3\ell m,\underline{k}} \sum u_{\ell,\underline{p}}\, u_{m,\underline{q}}$$

To do so, first note

$$M_{333,\underline{k}} = k_3 \left(1 - \frac{k_3^2}{k^2}\right) = k_3 \alpha_{\underline{k}}^2 \tag{23.a}$$

Denoting by b and c indices $\neq 3$,

$$M_{3b3,\underline{k}} = M_{33b,\underline{k}} = \frac{1}{2} k_3 \left(- \frac{k_3 k_b}{k^2}\right) + \frac{1}{2} k_b \left(1 - \frac{k_3^2}{k^2}\right)$$

$$= \frac{1}{2} k_b \left(\alpha_{\underline{k}}^2 - \beta_{\underline{k}}^2\right) \tag{23.b}$$

$$M_{3bc,\underline{k}} = \frac{1}{2} k_b \left(- \frac{k_3 k_c}{k^2}\right) + \frac{1}{2} k_c \left(- \frac{k_3 k_b}{k^2}\right)$$

$$= -\beta_{\underline{k}}\, k_b\, k_c\, k^{-1} \tag{23.c}$$

Using these we express some terms

$$M_{333,\underline{k}}\, u_{3,\underline{p}}\, u_{3,\underline{q}} = k_3 \alpha^2\, w_{\underline{p}}\, w_{\underline{q}} \tag{24.a}$$

$$M_{3b3,\underline{k}}\, u_{b,\underline{p}}\, u_{3,\underline{q}} \tag{24.b}$$

$$= \frac{1}{2} k_b \left(\alpha_{\underline{k}}^2 - \beta_{\underline{k}}^2\right) \left(\epsilon_{bj3}\, p_j\, v_{\underline{p}} - \sigma_{\underline{p}}\, p_b\, w_{\underline{p}}\right) w_{\underline{q}}\, p_h^{-1}$$

$$= \frac{1}{2} k_b \left(\alpha_{\underline{k}}^2 - \beta_{\underline{k}}^2\right) \left(\epsilon_{bj3} \frac{p_j}{p_h} v_{\underline{p}} w_{\underline{q}} - \sigma_{\underline{p}} \frac{p_b}{p_h} w_{\underline{p}} w_{\underline{q}}\right)$$

Likewise

$$M_{33b,\underline{k}}\, u_{3,\underline{p}}\, u_{b,\underline{q}} \tag{24.c}$$

$$= \frac{1}{2} k_b \left(\alpha_{\underline{k}}^2 - \beta_{\underline{k}}^2\right) \left(\varepsilon_{bj3} \frac{q_j}{q_h} v_{\underline{q}} w_{\underline{p}} - \sigma_{\underline{q}} \frac{q_b}{q_h} w_{\underline{q}} w_{\underline{p}}\right)$$

Next

$$M_{3bc,\underline{k}}\, u_{b,\underline{p}}\, u_{c,\underline{q}} \tag{24.d}$$

$$= -\beta_{\underline{k}} k_b k_c k^{-1} (\varepsilon_{bj3} p_j v_{\underline{p}} - p_b \sigma_{\underline{p}} w_{\underline{p}})$$

$$\cdot (\varepsilon_{cm3} q_m v_{\underline{q}} - q_c \sigma_{\underline{q}} w_{\underline{q}}) p_h^{-1} q_h^{-1}$$

$$= v_{\underline{p}} v_{\underline{q}} \frac{\beta_{\underline{k}}}{k\, p_h q_h} (k_b p_b k_c q_c - k_h^2 p_b q_b)$$

$$+ w_{\underline{p}} v_{\underline{q}} \frac{\beta_{\underline{k}}}{k p_h q_h} k_b k_c p_b \sigma_p \varepsilon_{cm3} q_m$$

$$+ v_{\underline{p}} w_{\underline{q}} \frac{\beta_{\underline{k}}}{k p_h q_h} k_b k_c \varepsilon_{bj3} p_j q_c \sigma_{\underline{q}}$$

$$- w_{\underline{p}} w_{\underline{q}} \frac{\beta_{\underline{k}}}{k p_h q_h} \sigma_{\underline{p}} \sigma_{\underline{q}} p_b q_c k_b k_c$$

Collecting terms, the right side of (13) is

$$-iM_{3\ell m,\underline{k}} \sum_{\underline{p}+\underline{q}=\underline{k}} u_{\ell,\underline{p}}\, u_{m,\underline{q}} \tag{25}$$

$$= -i \sum_{\underline{p}+\underline{q}=\underline{k}} \Big| C_1(\underline{k},\underline{p},\underline{q}) v_{\underline{p}} v_{\underline{q}} + C_2(\underline{k},\underline{p},\underline{q}) w_{\underline{p}} w_{\underline{q}}$$

$$+ C_3(\underline{k},\underline{p},\underline{q}) v_{\underline{p}} w_{\underline{q}} + C_3(\underline{k},\underline{q},\underline{p}) v_{\underline{q}} w_{\underline{p}} \Big|$$

where

$$C_1(\underline{k},\underline{p},\underline{q}) = \frac{\beta_{\underline{k}}}{kp_h\ q_h}(k_b p_b k_c q_c - k_h^{\ 2}\ p_b q_b)$$

$$= \frac{\beta_{\underline{k}}}{kp_h\ q_h}(\underline{e}^3 \cdot \underline{k} \times \underline{p})(\underline{e}^3 \cdot \underline{k} \times \underline{q})$$

$$C_2(\underline{k},\underline{p},\underline{q}) = k_3\alpha^2 - \frac{1}{2}k_b(\alpha_k^{\ 2} - \beta_k^{\ 2})\left(\frac{\sigma_p p_b}{p_h} + \frac{\sigma_q q_b}{q_h}\right)$$

$$- \frac{\beta_{\underline{k}}}{kp_h\ q_h}\ \sigma_{\underline{p}}\ \sigma_{\underline{q}}\ k_b\ k_c\ p_b\ q_c$$

$$C_3(\underline{k},\underline{p},\underline{q}) = \frac{1}{2}k_b(\alpha_{\underline{k}}^{\ 2} - \beta_{\underline{k}}^{\ 2})\epsilon_{bj3}\frac{p_j}{p_h}$$

$$+ \frac{\beta_{\underline{k}}}{kp_h\ q_h}k_b k_c\ \epsilon_{bj3}\ p_j\ q_c\ \sigma_{\underline{q}}$$

$$= \left[\frac{1}{2p_h}(\alpha_{\underline{k}}^{\ 2} - \beta_{\underline{k}}^{\ 2}) + \frac{\beta_{\underline{k}}}{kp_h\ q_h}k_c q_c \sigma_{\underline{q}}\right](\underline{e}^3 \cdot \underline{k} \times \underline{p})$$

For the right side of (14) we need

$$-i\epsilon_{sj3}\frac{k_j}{k_h}M_{s\ell m,\underline{k}}\sum_{\underline{p}+\underline{q}=\underline{k}}u_{\ell\underline{p}}\ u_{m\underline{q}}$$

Now $s \neq 3$ and again b and c denote indices $\neq 3$. Note that

$$M_{s33,\underline{k}} = k_3\left(-\frac{k_3 k_2}{k^2}\right) = -\ k_s\beta_{\underline{k}}^{\ 2} \tag{26.a}$$

$$M_{sb3,\underline{k}} = M_{s3b,\underline{k}}$$

$$= k_3 \left(\frac{1}{2}\delta_{sb} - \frac{k_s k_b}{k^2} \right) \tag{26.b}$$

$$M_{sbc,\underline{k}} = \frac{1}{2} \left(k_b \delta_{sc} + k_c \delta_{sb} - 2 \frac{k_b k_c k_s}{k^2}\right) \tag{26.c}$$

We begin evaluating.

$$\varepsilon_{sj3} \frac{k_j}{k_h} M_{s33,\underline{k}} u_{3\underline{p}} u_{3\underline{q}}$$

$$= (\varepsilon_{sj3} k_j k_s) \frac{-\delta_k^2}{k_h} w_{\underline{p}} w_{\underline{q}} = 0 \tag{27.a}$$

$$\varepsilon_{sj3} \frac{k_j}{k_h} M_{sb3,\underline{k}} u_{b\underline{p}} u_{3\underline{q}} \tag{27.b}$$

$$= \varepsilon_{sj3} \frac{k_j k_3}{k_h p_h} \left(\frac{1}{2} \delta_{sb} - \frac{k_s k_b}{k^2}\right)(\varepsilon_{b\ell 3} p_\ell v_{\underline{p}} - p_b \sigma_{\underline{p}} w_{\underline{p}}) w_{\underline{q}}$$

$$= \frac{p_b k_b k_3}{2k_h\, p_h} v_{\underline{p}} w_{\underline{q}} - \varepsilon_{sj3} \frac{p_s k_j k_3 \sigma_{\underline{p}}}{2k_h\, p_h} w_{\underline{p}}\, w_{\underline{q}}$$

Likewise

$$\varepsilon_{sj3} \frac{k_j}{k_h} M_{s3b,\underline{k}} u_{3\underline{p}} u_{b\underline{q}} \tag{27.c}$$

$$= \frac{q_b k_b k_3}{2k_h\, q_h} v_{\underline{q}} w_{\underline{p}} - \varepsilon_{sj3} \frac{q_s k_j k_3 \sigma_{\underline{q}}}{2k_h\, q_h} w_{\underline{p}} w_{\underline{q}}$$

Finally we develop as above and find

$$\varepsilon_{sj3} \frac{k_j}{k_h} M_{sbc,\underline{k}}\, u_{b\underline{p}}\, u_{c\underline{q}} \tag{27.d}$$

$$= v_{\underline{p}} v_{\underline{q}} (2k_h p_h q_h)^{-1} \Big[\varepsilon_{cm3} (k_c q_m p_b k_b$$

$$+ k_c q_m p_j k_j - k_h^2 p_c q_m) \Big]$$

$$+ w_{\underline{p}} w_{\underline{q}} \sigma_{\underline{p}} \sigma_{\underline{q}} (2k_h p_h q_h)^{-1} \Big[\varepsilon_{sj3} (p_s k_j k_c q_c$$

$$+ q_s k_j k_b p_b) \Big]$$

$$- v_{\underline{p}} w_{\underline{q}} \sigma_{\underline{q}} (2k_h p_h q_h)^{-1} \Big[p_b k_b q_c k_c$$

$$+ k_c q_c k_j p_j - k_h^2 p_c q_c \Big]$$

$$- v_{\underline{q}} w_{\underline{p}} \cdot (p,q \text{ sym. coef.})$$

Collecting terms, the right side of (14) is

$$-i\varepsilon_{sj3} \frac{k_j}{k_h} M_{s\ell m,\underline{k}} \sum_{\underline{p}+\underline{q}=\underline{k}} u_{\ell\underline{p}}\, u_{m\underline{q}} \tag{28}$$

$$= -i \sum_{\underline{p}+\underline{q}=\underline{k}} \Big\{ C_4(\underline{k},\underline{p},\underline{q}) v_{\underline{p}} v_{\underline{q}} + C_5(\underline{k},\underline{p},\underline{q}) w_{\underline{p}} w_{\underline{q}}$$

$$+ C_6(\underline{k},\underline{p},\underline{q}) v_{\underline{p}} w_{\underline{q}} + C_6(\underline{k},\underline{q},\underline{p}) v_{\underline{q}} w_{\underline{p}} \Big\}$$

where

$$C_4(\underline{k},\underline{p},\underline{q}) = \frac{\varepsilon_{cm3}}{2k_h\ p_h\ q_h} \left[2\ p_b k_b k_c q_m - k_h^{\ 2} p_c q_m \right]$$

$$C_5(\underline{k},\underline{p},\underline{q}) = \frac{\varepsilon_{sj3}}{2k_h\ p_h\ q_h} \Big[p_s k_j (\sigma_{\underline{p}} \sigma_{\underline{q}} k_c q_c - \sigma_{\underline{p}} k_3 q_h)$$

$$+ q_s k_j (\sigma_{\underline{p}} \sigma_{\underline{q}} k_b p_b - \sigma_{\underline{q}} k_3 p_h) \Big]$$

$$C_6(\underline{k},\underline{p},\underline{q}) = (2k_h p_h q_h)^{-1} \Big[k_3 q_h p_b k_b - 2\ \sigma_{\underline{q}} k_b p_b k_c q q_c$$

$$+ \sigma_{\underline{q}} k_h^{\ 2} p_c q_c \Big]$$

The right side of (15) is more easily seen:

$$-ik_j \sum_{\underline{p}+\underline{q}=\underline{k}} u_{j\underline{p}} \rho_{\underline{q}} \tag{29}$$

$$= -i \sum_{\underline{p}+\underline{q}=\underline{k}} k_s (\varepsilon_{sj3} p_j v_{\underline{p}} - p_s \sigma_{\underline{p}} w_{\underline{p}}) \rho_{\underline{q}} / p_h + k_3 w_{\underline{p}} \rho_{\underline{q}}$$

$$= -i \sum_{\underline{p}+\underline{q}=\underline{k}} v_{\underline{p}} \rho_{\underline{q}}\ \varepsilon_{sj3} \frac{k_s p_j}{p_h} - w_{\underline{p}} \rho_{\underline{q}} \left[\sigma_{\underline{p}} \frac{k_s p_s}{p_h} - k_3 \right]$$

Together, (25), (28) and (29) give the complete right side of (16). In matrix notation, that right side may be written

$$- i \sum_{\underline{p}+\underline{q}=\underline{k}} \underline{\underline{N}}_{\underline{kpq}} : \underline{y}_{\underline{p}} \, \underline{y}_{\underline{q}}$$

where $\underline{\underline{N}}$ is a 3 x 3 x 3 matrix whose elements are functions of $\underline{k}$, $\underline{p}$ and $\underline{q}$. For each $\underline{k}$, $\underline{p}$ and $\underline{q}$ there are 27 elements of $\underline{\underline{N}}$. However, only ten are different from zero. Explicitly these are

$$N_{111} = C_2(\underline{k},\underline{p},\underline{q})$$

$$N_{112} = C_3(\underline{k},\underline{p},\underline{q})$$

$$N_{121} = C_3(\underline{k},\underline{q},\underline{p})$$

$$N_{122} = C_1(\underline{k},\underline{p},\underline{q})$$

$$N_{211} = C_5(\underline{k},\underline{p},\underline{q})$$

$$N_{212} = C_6(\underline{k},\underline{p},\underline{q})$$

$$N_{221} = C_6(\underline{k},\underline{q},\underline{p})$$

$$N_{222} = C_4(\underline{k},\underline{p},\underline{q})$$

$$N_{331} = k_3 \quad \sigma_{\underline{p}} \frac{k_s p_s}{p_h}$$

$$N_{332} = \varepsilon_{sj3} \frac{k_s p_j}{p_h}$$

In like fashion was seek to develop the interaction equations among a^+, a^- and h . The straightforward but frightfully tedious method is to make the substitution (20) throughout (16).

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Fourier representations of the equations of motion for a rotating stratified fluid are obtained both in the two-dimensional and three-dimensional wave number space. It is shown that these spectral equations can be further transformed to reveal explicitly the interaction between waves and waves and between waves and eddies by decomposing the Fourier coefficients in terms of three basis functions, that correspond to the "upward" propagating and the "downward" propagating wave modes and the zero frequency horizontal eddy mode in the linear analysis. In the case of two-dimensional rotating stratified fluid motion, the transformation is made and full spectral equations up to second moments for these basis functions are derived.

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